1. ZT, Varianta A

1. Označme jevy A_i ... otázka je z i-tého okruhu, $i \in \{1,2,3\}$

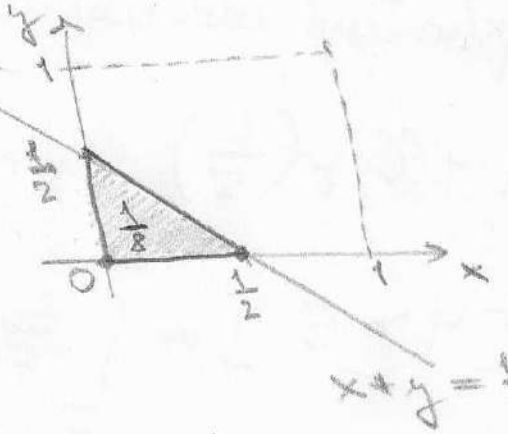
B ... student je připraven na otázku

$$P(B) = \sum_{i=1}^{3} P(B|A_i) \cdot P(A_i) = \frac{2}{5} \cdot \frac{5}{25} + \frac{3}{10} \cdot \frac{10}{25} + \frac{5}{10} \cdot \frac{10}{25} =$$

$$= \frac{2}{25} + \frac{3}{25} + \frac{5}{25} = \frac{10}{25} = \underline{\underline{0,4}}$$

2. Náhodně vybereme dvě čísla $x, y \in (0,1)$, jaká je pravděpodobnost, že $x + y < \frac{1}{2}$?

$$P\left[x+y<\frac{1}{2}\right] = \frac{\text{obsah trojúhelníku}}{\text{obsah čtverce}} = \frac{\frac{1}{8}}{1} = \underline{\underline{\frac{1}{8}}}$$



3. $P(\text{pasažér se dostaví k letu}) = 0,97$

X_i ... i-tý pasažér se dostaví k letu, $i \in \{1, \dots, 100\}$

$P[X_i = 1] = 0,97$ $\qquad X_i \sim Alt(0,97)$

$P[X_i = 0] = 0,03$

Y ... počet lidí, kteří se k letu dostaví: $Y = \sum_{i=1}^{100} X_i$

a) $P[Y > 98] = P[Y = 99] + P[Y = 100] = \binom{100}{99} \cdot 0,97^{99} \cdot 0,03 + 0,97^{100}$

b) $P[Y = k] = \binom{100}{k} \cdot 0,97^{k} \cdot 0,03^{100-k}$ $\qquad$ BINOMICKÉ ROZDĚLENÍ

$k \in \{0, \dots, 100\}$ $\qquad Y \sim Bi(100, 0,97)$

4\. X... počet zákazníků, kteří přijdou během 5 minut

– Poissonovo rozdělení

s parametrem $\lambda = \frac{5}{60} \cdot 24 = 2$

$$P[X=k] = \frac{2^k}{k!} \cdot e^{-2}, \quad k \in \mathbb{N}_0$$

$$P[X=0] = \underline{\underline{e^{-2}}}$$

jinak: Y... doba čekání na příchod zákazníka

– Exponenciální rozdělení s parametrem $\lambda = \frac{1}{EX}$

průměrná doba čekání na zákazníka je $\frac{60}{24} = \frac{5}{2}$ min.

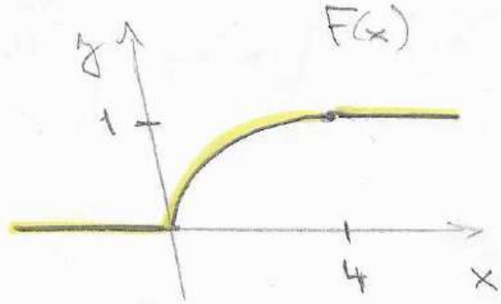
$$Y \sim Exp\left(\frac{2}{5}\right), \text{ tedy } f_Y(x) = \begin{cases} \frac{2}{5} e^{-\frac{2}{5}x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

$$P[Y > 5] = \int_5^\infty \frac{2}{5} e^{-\frac{2}{5}x}\, dx = \left[-e^{-\frac{2}{5}x}\right]_5^\infty =$$

$$= \underbrace{\lim_{x \to \infty}\left(-e^{-\frac{2}{5}x}\right)}_{0} + e^{-2} = \underline{\underline{e^{-2}}}$$

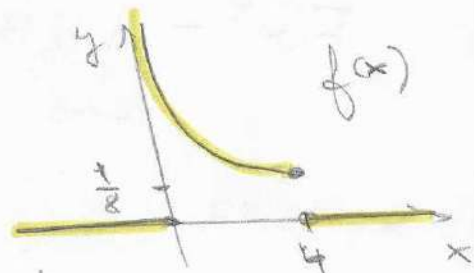
5.

$$F(x) = \begin{cases} 0 & , x<0 \\ \frac{1}{2}\sqrt{x} & , x\in[0,4] \\ 1 & , x>4 \end{cases}$$



hustota:

$$f(x) = \begin{cases} \frac{1}{4}\cdot\frac{1}{\sqrt{x}} & , x\in(0,4] \\ 0 & , x\notin[0,4] \end{cases}$$



- $EX = \int_{\mathbb{R}} x f(x)\,dx = \int_0^4 x\cdot\frac{1}{4\sqrt{x}}\,dx = \frac{1}{4}\int_0^4 x^{\frac{1}{2}}\,dx = \frac{1}{4}\cdot\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$

$= \frac{1}{4}\cdot\left(\frac{2}{3}\cdot 4^{\frac{3}{2}} - 0\right) = \frac{1}{6}\cdot 2^3 = \frac{8}{6} = \frac{4}{3}$

- $EX^2 = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^4 x^2\cdot\frac{1}{4\sqrt{x}}\,dx = \frac{1}{4}\cdot\int_0^4 x^{\frac{3}{2}}\,dx =$

$= \frac{1}{4}\cdot\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^4 = \frac{1}{10}\cdot 4^{\frac{5}{2}} = \frac{1}{10}\cdot 2^5 = \frac{32}{10} = \frac{16}{5}$

- Rozptyl: $\operatorname{var}X = EX^2 - (EX)^2 = \frac{16}{5} - \left(\frac{4}{3}\right)^2 =$

$= \frac{16}{5} - \frac{16}{9} = 16\cdot\underbrace{\left(\frac{1}{5}-\frac{1}{9}\right)}_{\frac{9-5}{45}} = 16\cdot\frac{4}{45} = \frac{64}{45}$

- $P\left[1\le X\le \frac{9}{4}\right] = F\left(\frac{9}{4}\right) - F(1) =$

$= \frac{1}{2}\cdot\sqrt{\frac{9}{4}} - \frac{1}{2}\cdot\sqrt{1} = \frac{1}{2}\underbrace{\left(\frac{3}{2}-1\right)}_{\frac{1}{2}} = \frac{1}{4}$

⑥ $f(x) = \begin{cases} \frac{15}{16} \cdot \sqrt{x} \cdot (x+1) & , x \in [0,1] \\ 0 & , x \notin [0,1] \end{cases}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & , x < 0 \\ \int_0^x f(t)\,dt & , x \in [0,1] \\ 1 & , x \geq 1 \end{cases}$$

pro $x \in [0,1]$:

$$F(x) = \int_0^x \frac{15}{16} \sqrt{t}(t+1)\,dt = \frac{15}{16} \cdot \int_0^x \left(t^{\frac{3}{2}} + t^{\frac{1}{2}} \right) dt =$$

$$= \frac{15}{16} \cdot \left[\frac{2}{5} t^{\frac{5}{2}} + \frac{2}{3} t^{\frac{3}{2}} \right]_0^x = \frac{3}{8} x^{\frac{5}{2}} + \frac{5}{8} x^{\frac{3}{2}}$$

$$EX = \int_{\mathbb{R}} x f(x)\,dx = \frac{15}{16} \int_0^1 x \cdot \sqrt{x}(x+1)\,dx = \frac{15}{16} \cdot \int_0^1 \left(x^{\frac{5}{2}} + x^{\frac{3}{2}} \right)$$

$$= \frac{15}{16} \cdot \left[\frac{2}{7} x^{\frac{7}{2}} + \frac{2}{5} x^{\frac{5}{2}} \right]_0^1 = \frac{15}{16} \cdot \underbrace{\left(\frac{2}{7} + \frac{2}{5} \right)}_{2 \cdot \frac{5+7}{35}} = \frac{\cancel{15}^{3}}{\cancel{16}_{2}} \cdot \frac{\cancel{24}^{3}}{\cancel{35}_{7}}$$

$$= \frac{9}{14}$$

$$EX^2 = \int_{\mathbb{R}} x^2 f(x)\,dx = \frac{15}{16} \cdot \int_0^1 \underbrace{\left(x^2 \cdot \sqrt{x}(x+1) \right)}_{x^{\frac{7}{2}} + x^{\frac{5}{2}}} dx = \frac{15}{16} \left[\frac{2}{9} x^{\frac{9}{2}} + \frac{2}{7} x^{\frac{7}{2}} \right]$$

$$= \frac{15}{16} \cdot \left(\frac{2}{9} + \frac{2}{7} \right) = \frac{15}{\cancel{8}} \cdot \frac{\cancel{16}^{2}}{63} = \frac{30}{63} = \frac{10}{21}$$

$$\mathrm{var}\,X = EX^2 - (EX)^2 = \frac{10}{21} - \left(\frac{9}{14} \right)^2 = \frac{10}{21} - \frac{81}{14^2} = \frac{37}{588}$$

$\doteq 0{,}06$