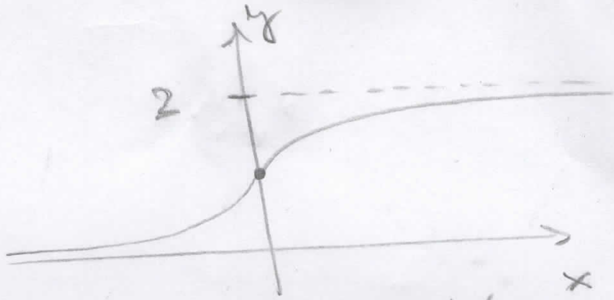
Variantа A, 1. ZT

(1.) $f(x) = \begin{cases} e^{x}, & x<0 \\ 2-e^{-x}, & x \geq 0 \end{cases}$



$H_f = (0,2)$

$\forall a,b \in D_f,\ a \neq b : f(a) \neq f(b) \Rightarrow$ funkce je prostá

$\lim\limits_{x \to 0_-} f(x) = e^0 = 1 = f(0) \Rightarrow$ spojitá v bodě $x=0$

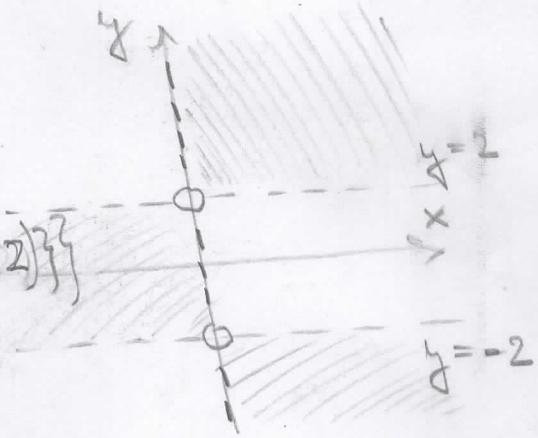
$f'(x) = \begin{cases} e^{x}, & x<0 \\ e^{-x}, & x \geq 0 \end{cases}$ $\lim\limits_{x \to 0_+} f'(x) = \lim\limits_{x \to 0_-} f'(x)$

$\Rightarrow$ funkce je diferencovatelná

(2.) $f(x,y) = \ln(xy^2 - 4x)$

$xy^2 - 4x = x(y-2)(y+2) > 0$

$D_f = \{(-\infty,0) \times (-2,2) \cup (0,\infty) \times \{\mathbb{R} \setminus (-2,2)\}\}$



(3.) $f(x,y) = e^{6x} \cdot (y^2 + 3x + 4y + 3)$

$\frac{\partial f}{\partial x} = 6e^{6x} \cdot (y^2 + 3x + 4y + 3) + 3e^{6x}$

$= 3e^{6x} \cdot (2y^2 + 6x + 8y + 7) = 0$ dosazení

$\frac{\partial f}{\partial y} = e^{6x} \cdot (2y + 4) = 0 \iff \boxed{y = -2}$

$8 + 6x - 16 + 7 = 0$

$6x = 1$

$\boxed{x = \frac{1}{6}}$

stacionární bod

$\left[\frac{1}{6}; -2\right]$

$$\frac{\partial^2 f}{\partial x^2} = 18e^{6x}\cdot(2y^2+6x+8y+7)+3e^{6x}\cdot 6$$

$$= 18e^{6x}\cdot(2y^2+6x+8y+8)\Big|_{[\frac{1}{6};-2]} = 18e$$

$$\frac{\partial^2 f}{\partial y^2} = 2e^{6x}\Big|_{[\frac{1}{6};-2]} = 2e$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = 6e^{6x}\cdot(2y+4)\Big|_{[\frac{1}{6};-2]} = 0$$

Hessova matice: $H(\frac{1}{6},-2) = \begin{pmatrix} 18e & 0 \\ 0 & 2e \end{pmatrix}$

$\frac{\partial^2 f}{\partial x^2}(\frac{1}{6},-2) > 0 \wedge \det H > 0 \implies [\frac{1}{6};-2]$ je lok. minimum

neboť Hessova matice je pozitivně definitní

(4.) $f(x,y) = \frac{y^2}{1-x}\Big|_{[0,1]} = 1$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= +\frac{y^2}{(1-x)^2}\Big|_{[0,1]} = 1 \\ \frac{\partial f}{\partial y} &= \frac{2y}{1-x}\Big|_{[0,1]} = 2 \end{aligned}\right\} \nabla f(0,1) = (1,2)$$

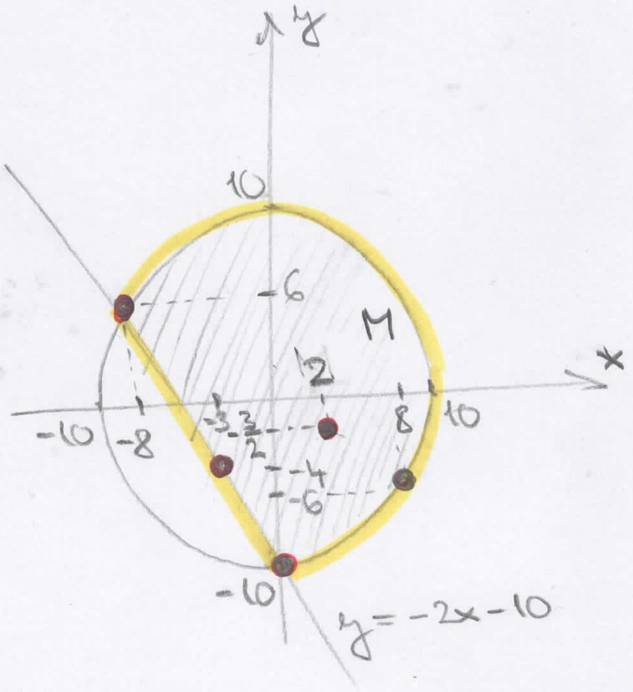
Tečná rovina: $\boxed{z-1 = 1\cdot(x-0)+2\cdot(y-1)}$

5. $f(x,y) = x^2+y^2-4x+3y$

$M = \{(x,y)\in\mathbb{R}^2;\ x^2+y^2\le 100;\ 2x+y+10\ge 0\}$

průsečíky přímky a kružnice:

$$\begin{aligned} x^2+y^2 &= 100 \\ y &= -2x-10 \end{aligned}$$ ← dosadíme



$x^2+(-2x-10)^2 = 100$

$x^2+4x^2+40x+100 = 100$

$5x^2+40x = 0$

$5x(x+8) = 0$

$x=0 \quad \vee \quad x=-8$

$y=-10 \qquad y=6$

1. Volné extrémy: $\frac{\partial f}{\partial x} = 2x-4 = 0 \iff x=2$

$\frac{\partial f}{\partial y} = 2y+3 = 0 \iff y=-\frac{3}{2}$

stacionární bod $[2;-\frac{3}{2}]\in M$

2. Vázané extrémy: I. úsečka $y=-2x-10$, $x\in\langle -8,0\rangle$

$g(x) := f(x,-2x-10) = x^2+(-2x-10)^2-4x+3\cdot(-2x-10) =$

$= 5x^2+30x+70$

$g'(x) = 10x+30 = 0 \iff x=-3 \Rightarrow$ kandidát $[-3;-4]$

II. oblouk $\underbrace{x^2+y^2-100}_{=:g(x,y)} = 0$

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x-4 & 2y+3 \\ 2x & 2y \end{vmatrix} = (2x-4)\cdot 2y - 2x\cdot(2y+3) =$$

$$= \cancel{4xy}-8y-\cancel{4xy}-6x = 0 \iff y = -\frac{3}{4}x$$

$x^2+\left(-\frac{3}{4}x\right)^2-100 = 0$

$x^2+\frac{9}{16}x^2 = 100$

$\frac{25}{16}x^2 = 100 \Rightarrow x^2=64 \Rightarrow x=\pm 8,\ y=\mp 6$

kandidáti $[8,-6]$ $[-8,6]$

kandidáti na extrém:

$[2; -\frac{3}{2}]$ $\quad f(2, -\frac{3}{2}) = 4 + \frac{9}{4} - 8 - \frac{9}{2} = -\frac{25}{4}$ minimum

$[-3; -4]$ $\quad f(-3, -4) = 9 + 16 + 12 - 12 = 25$

$[8; -6]$ $\quad f(8, -6) = 64 + 36 - 32 - 18 = 50$

$[-8; 6]$ $\quad f(-8, 6) = 64 + 36 + 32 + 18 = 150$ maximum

$[0; -10]$ $\quad f(0, -10) = 100 - 30 = 70$